Differentiation from first principles using spreadsheets

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INTRODUCTION

Students, when they are introduced to differential calculus in Year 11, often have a number of difficulties. There is the difficulty of understanding what a limit is (Juter, 2006; Tall, 1993). Many students have difficulty seeing a tangent as the limiting case of a secant (Orton, 1983; Ryan, 1992). The algebraic manipulations required in differentiation from first principles, often obscure the process and principles involved. Finally, the rules for differentiation are often presented as a list to be rote-learned (e.g., Nolan et al., 2006), which is contrary to good educational practice (McInerney & McInerney, 2002).

For example, when differentiating the function, $f(x) = x^3$, students have to worry about factoring cubic equations and simplifying rational polynomials, in addition to understanding the fundamental aspect of determining the gradient of a secant as it approaches a tangent (Orton, 1983; Ryan, 1992).

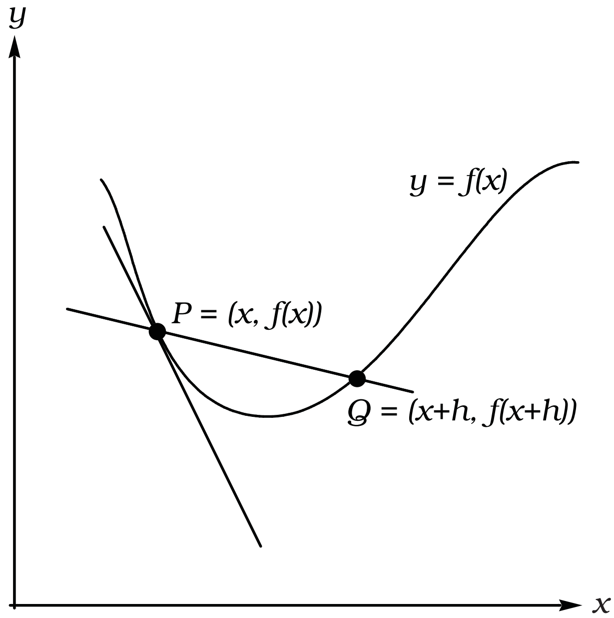

Figure 1. The gradient of a function, y=f(x), at point P is found as the limiting value of the gradient of the secant PQ as the point Q approaches point P.

Formally, this involves evaluating the slope of a secant defined by two points *P* and *Q* on the curve, $y = f(x)$:

$$\begin{aligned}\text{gradient of secant} &= \frac{\Delta y}{\Delta x} \\ &= \frac{f(x+h)-f(x)}{h} \qquad (1)\end{aligned}$$

where $(x, f(x))$ and $(x+h, f(x+h))$ are the (x,y) coordinates of the points *P* and *Q*. Algebraically, one then takes the limit as the secant tends to a tangent (i.e., the two points *P* and *Q* approach each other).

$$\begin{aligned}\text{gradient of tangent} &= \lim_{h\to 0} \text{gradient of secant} \\ &= \lim_{h\to 0}\frac{\Delta y}{\Delta x} \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad (2)\end{aligned}$$

After laboriously applying Equation (2) to a number of examples involving different functions, $f(x)$, students are instructed that the rules in Table 1 will enable them to differentiate any polynomial function (e.g., Nolan et al., 2006).

Table 1. Rules for finding the derivative of polynomial functions.

	function	derivative or gradient
	$f(x) = x^n$	$f'(x) = x^{n-1}$
multiplying by a constant, a	$f(x) = ax^n$	$f'(x) = nax^{n-1}$
c is a constant	$f(x) = c$	$f'(x) = 0$
sum of two functions	$f(x) = g(x) + h(x)$	$f'(x) = g'(x) + h'(x)$

Memorisation of rules, such as Table 1, is contrary to good educational practice (McInerney & McInerney, 1998).

In the teaching of calculus, the algebraic derivation of the derivative (gradient function) enables the student to obtain an analytic *global* gradient function. However, to the best of the author's knowledge, all current technology-based approaches require the student to obtain the derivative (gradient) at a single point by implementing differentiation using first principles. The following sections show that the ability of spreadsheets to fit a polynomial to a set of discrete (x,y) points, enables students to not just evaluate a gradient at a single point, but at a whole family of points, thus generating the analytic global gradient function of secants without doing any algebraic manipulations. Students can then perform "numerical experiments" to see the effect of taking the limit as the secants tend to tangents of the original function. Finally, students can derive the rules for differentiation through exploration and experimentation, again, without doing any algebraic manipulations. This approach enables the class to focus on the concepts being taught, rather than being hindered by the mechanics of (for example) trying to factorise a cubic polynomial.

Using a spreadsheet for differentiation by first principles

Even 10 years ago, most students at the end of junior secondary school (Year 10) were able to use spreadsheets (Meredyth et al., 1999). Hence this paper assumes that students are familiar with the use of spreadsheets, but expertise is not required for the following.

Using a spreadsheet, students are asked to create a table of x- and y-values, which are the independent variable and the function values, respectively. In the example depicted in Figure 2, column D of the spreadsheet lists values for the function, $f(x) = x^2$.

	A	B	C	D	E	F	G
1			x	f(x)	x+h	f(x+h)	slope(x)
2			-5	25	-3.5	12.25	-8.5
3	delta x =	0.1	-4.9	24.01	-3.4	11.56	-8.3
4			-4.8	23.04	-3.3	10.89	-8.1
5	h =	1.5	-4.7	22.09	-3.2	10.24	-7.9
6			-4.6	21.16	-3.1	9.61	-7.7
7			-4.5	20.25	-3	9	-7.5
8			-4.4	19.36	-2.9	8.41	-7.3
9			-4.3	18.49	-2.8	7.84	-7.1
10			-4.2	17.64	-2.7	7.29	-6.9
11			-4.1	16.81	-2.6	6.76	-6.7
12			-4	16	-2.5	6.25	-6.5
13			-3.9	15.21	-2.4	5.76	-6.3
14			-3.8	14.44	-2.3	5.29	-6.1
15			-3.7	13.69	-2.2	4.84	-5.9
16			-3.6	12.96	-2.1	4.41	-5.7
17			-3.5	12.25	-2	4	-5.5
18			-3.4	11.56	-1.9	3.61	-5.3
19			-3.3	10.89	-1.8	3.24	-5.1
20			-3.2	10.24	-1.7	2.89	-4.9
21			-3.1	9.61	-1.6	2.56	-4.7
22			-3	9	-1.5	2.25	-4.5
23			-2.9	8.41	-1.4	1.96	-4.3
24			-2.8	7.84	-1.3	1.69	-4.1
25			-2.7	7.29	-1.2	1.44	-3.9
26			-2.6	6.76	-1.1	1.21	-3.7
27			-2.5	6.25	-1	1	-3.5
28			-2.4	5.76	-0.9	0.81	-3.3

Figure 2. Part of a spreadsheet showing tabulated values of x (Column C), $f(x) = x^2$ (column D), x+h (Column E), $f(x+h) = (x=h)^2$ (Column F), and the slope of the secant defined by Equation (1) (Column G). Cell B5 contains the value of h.

Students can plot the curve, $y = f(x)$, and use the curve of best fit ("Add trendline") to confirm that the plotted function is indeed $f(x) = x^2$ as shown in Figure 3.

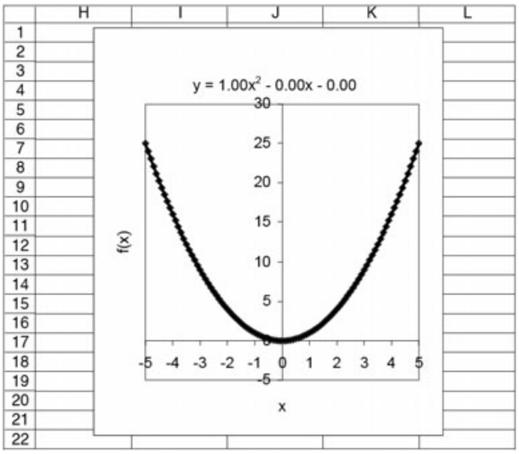

Figure 3. Part of a spreadsheet showing a plot of the (x,y) values depicted in Figure 2 (Columns C and D).

For each point, $P \equiv (x, f(x))$ (Columns C and D in Figure 2), students can determine the associated point, $Q \equiv (x+h, f(x+h))$ (Columns E and F in Figure 2). Next, the slope of the segment or secant, PQ, can be determined using Equation (1) as shown in Column G in Figure 2. There have been several reports of how to use a spreadsheet or similar programs to determine the gradient of a single secant such as that in Figure 1 (e.g., Bloom et al., 1987; Nolan et al., 2006), but instead of looking at a single secant, it is possible to investigate the gradient function of a whole family of secants as defined by the spreadsheet in Figure 2. This gradient function (Equation 1 and Column G in Figure 2) is plotted in Figure 4. The curve of best fit has been used to find that the gradient function has the formula:

$$\text{secant gradient function} = 2x + 1.5, \text{ for } h = 1.5 \qquad (3)$$

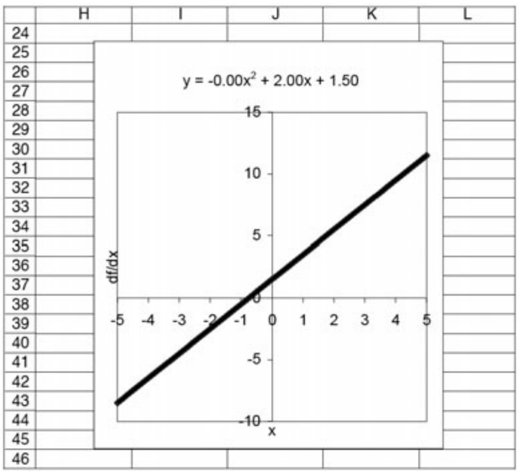

Figure 4. Part of a spreadsheet showing a plot of gradient function (Equation 1 and Column G in Figure 2).

At this stage, students can perform numerical experimentation ("what if" scenarios). They can see what happens to the x and $x+h$ values (Columns C

and E in Figure 2) as the value of h is made smaller. There are no restrictions on the value of h so it is possible to compare the limit from above ($h > 0$) with the limit from below ($h < 0$). They can see what happens to the values of the $f(x)$ and $f(x+h)$ functions (Columns D and F in Figure 2) as the value of h is made smaller. Finally, they can see the behaviour of the gradient function as the value of h is made smaller.

Each row in the spreadsheet in above procedure has implemented finding the derivative at a point by taking successively smaller values of h for a single value of x and noticing that eventually the slope is the same (i.e., has a limit). By repeating this for many points (on several rows of the table), the procedure finds the derivative at multiple points simultaneously, thus finding the entire gradient function.

Students can test the effect of changing the function, $f(x)$, and thus experimentally derive the rules listed in Table 1. Note that this is an experimental proof, which can then be followed by a formal mathematical derivation of the rules.

Discussion

There have been several reports of how to use a spreadsheet or similar programs to determine the gradient of a secant such as that in Figure 1 (e.g., Bloom et al., 1987; Nolan et al., 2006). This paper extends that idea to investigate the gradient function of a whole family of secants.

After work was begun on this paper, Ubuz (2007) reported the effectiveness of students using a specialist software to investigate the relationships between function and their gradients, which is very similar to the concept of this paper. However, the use of any commonly available spreadsheet program means that teachers and their students are not dependent on any specialist software, nor hindered by the extra learning hurdle presented by learning yet another program (Galbraith & Pemberton, 2002). Spreadsheets, like other "worldware" (Ehrmann, 1995), are commonly available and inexpensive because of the large user base, and relatively easy to learn. The choice of a spreadsheet also means that the exercise can be run on a number of platforms: spreadsheets are even finding their way onto hand-held personal digital assistants (PDAs)!

The activities outlined in the previous section, allows students to quickly perform a large number of numerical operations and thus to see the effect of taking limits and seeing how as $h \rightarrow 0$, the gradient of the secant heads to a limiting value, instead of concentrating on the mechanics of factorising polynomial expressions (Day, 1986; Solow, 1994). The value of simulations has been discussed by various authors (Haile, 1992; Neuwirth & Arganbright, 2004). Simulations are a form of mathematical experimentation (Hénon & Heiles, 1964), which will help learners who favour tactile–kinesthetic learning preferences (Dunn et al., 1995; Goodwin & Smith, 2003).

Kolb (1984), in a study of average learning style inventory (LSI) scores, found that university-level mathematics students had a significant tendency

towards abstract conceptualisation — much more so than average students who tended favour concrete experience more than the self-selected mathematics students. While there is some element of self-selection in secondary schools, many students study Year 11 mathematics because it is required for future studies or because it will help achieve higher Year 12 marks and university entrance rankings. These students do not truly have an interest in or an affinity for mathematics. Kolb's work implies that these students would learn better with more concrete experience learning activities. Most Year 11 textbooks first explain the mathematical concepts, and then permit students to work through examples and exercises. In this proposed activity, students experiment with a spreadsheet (a concrete experience learning activity and then use their results to construct the "rules" associated with differentiation. Students learn better when they actively construct meaning (McInerney & McInerney, 1998). Barnes (1995) also suggests that students should initially explore the concepts associated with finding gradients, gradients of secants, etc., but with minimal use of calculators and computers: a similar idea of exploration is suggested in this paper, but the difference is that here technology is used to accelerate the process.

The use of spreadsheets will not replace the formal proofs and rigour of the traditional approach to teaching and learning differential calculus. Instead, by combining use of tables of values, graphs and numerical experimentation, teachers will enable concrete learners, not just abstract thinkers to experience the behaviour of derivatives. This follows a long tradition advocated by Pólya (1962, 1971) and others (Goldenberg, 1995; Kaput, 1992).

An additional benefit is that this activity will enhance the ability of students to use spreadsheets, which is a skill desired by employers (The Australian Chamber of Commerce and Industry & the Business Council of Australia, 2002). Spreadsheets can also be used to empower students, who are lacking confidence in their mathematical ability, in their study of other disciplines (Lim, 2003a, 2003b, 2003c, 2005, 2006, 2008; Lim & Coleman, 2005). Since the use of spreadsheets is a tool to aid learning, not an end in itself, teachers who wish to use the spreadsheet document as a 'blackbox' application, can obtain the spreadsheet outlined in this paper from the author.

Conclusion

Many students have difficulty learning about differentiation. Other authors have used spreadsheets to find the gradient of a secant. Here it is shown that spreadsheets can quickly evaluate the gradient function of a whole family of secants. This can be done numerically, and then by using the curve of best fit, to find the equation of the secants' gradient function. Students can perform numerical experiments to find limits and to find the actual gradient function (i.e., the derivative) as a limiting function as a parameter, h, becomes smaller. Other numerical experiments can experimentally derive the usual rules for finding the derivative of polynomial functions. These experimental proofs can be used to introduce a more formal approach to the differential calculus.

This approach is a paradigm shift in the teaching of calculus from one that is firmly based in algebraic manipulation to one that uses technology for exploration and experimentation.

The spreadsheet outlined in this paper, sample teacher notes and a sample student handout can be obtained from the author at <lim@deakin.edu.au>.

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